Answers To a Selection of Problems from

Classical Electrodynamics

John David Jackson

by Kasper van Wijk

Center for Wave Phenomena
Department of Geophysics
Colorado School of Mines
Golden, CO 80401

Samizdat
Press

Published by the Samizdat Press

Center for Wave Phenomena
Department of Geophysics
Colorado School of Mines
Golden, Colorado 80401

and
New England Research
76 Olcott Drive
White River Junction, Vermont 05001

Release 2.0, January 1999

Contents

Introduction

This is a collection of my answers to problems from a graduate course in electrodynamics. These problems are mainly from the book by Jackson [3], but appended are some practice problems. My answers are by no means guaranteed to be perfect, but I hope they will provide the reader with a guideline to understand the problems.

Throughout these notes I will refer to equations and pages of Jackson and Duffin [2]. The latter is a textbook in electricity and magnetism that I used as an undergraduate student. References to equations starting with a “D” are from the book by Duffin. Accordingly, equations starting with the letter “J” refer to Jackson.

In general, primed variables denote vectors or components of vectors related to the distance between source and origin. Unprimed coordinates refer to the location of the point of interest.

The text will be a work in progress. As time progresses, I will add more chapters.

Chapter 1

Introduction to Electrostatics

1.1 Electric Fields for a Hollow Conductor

a. The Location of Free Charges in the Conductor

Gauss' law states:

$$\frac{\rho}{\epsilon_0} = \nabla \cdot \mathbf{E}, \tag{1.1}$$

where ρ is the volume charge density and ϵ_0 is the permittivity of free space. We know that conductors allow charges free to move within. So, when placed in an external static electric field charges move to the surface of the conductor, canceling the external field inside the conductor. Therefore, a conductor carrying only static charge can have no electric field within its material, which means the volume charge density is zero and excess charges lie on the surface of a conductor.

b. The Electric Field inside a Hollow Conductor

When the free charge lies outside the cavity circumferenced by conducting material (see figure 1.1a), Gauss' law simplifies to Laplace's equation in the cavity. The conducting material forms a volume of equipotential, because the electric field in the conductor is zero and

$$\mathbf{E} = -\nabla\Phi \tag{1.2}$$

Since the potential is a continuous function across a charged boundary, the potential on the inner surface of the conductor has to be constant. This is now a problem satisfying Laplace's equation with Dirichlet boundary conditions. In section 1.9 of Jackson, it is shown that the solution for this problem is unique. The constant value of the potential on the outer surface of the cavity satisfies Laplace's equation and is therefore *the* solution. In other words, the hollow conductor acts like a electric field shield for the cavity.

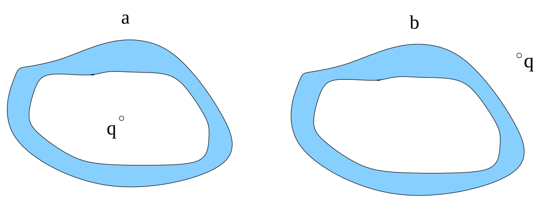

Figure 1.1: *a: point charge in the cavity of a hollow conductor. b: point charge outside the cavity of a hollow conductor.*

With a point charge q *inside* the cavity (see figure 1.1b), we use the following representation of Gauss' law:

$$\oint \mathbf{E} \cdot d\mathbf{S} = \frac{q}{\epsilon_0} \tag{1.3}$$

Therefore, the electric field inside the hollow conductor is non-zero. Note: the electric field outside the conductor due to a point source inside is influenced by the shape of the conductor, as you can in the next problem.

c. The Direction of the Electric Field outside a Conductor

An electrostatic field is conservative. Therefore, the circulation of **E** around any closed path is zero

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \tag{1.4}$$

This is called the circuital law for **E** (E4.14 or J1.21). I have drawn a closed path in four legs

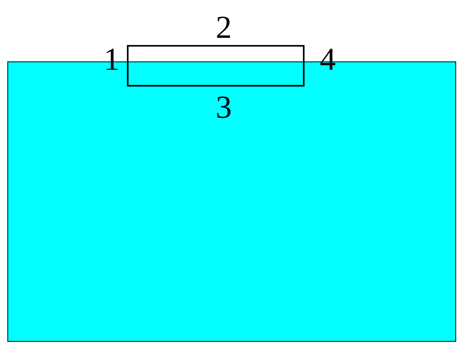

Figure 1.2: *Electric field near the surface of a charged spherical conductor. A closed path crossing the surface of the conductor is divided in four sections.*

through the surface of a rectangular conductor (figure 1.2). Sections 1 and 4 can be chosen negligible small. Also, we have seen earlier that the field in the conductor (section 3) is zero. For the total integral around the closed path to be zero, the tangential component (section 2) has to be zero. Therefore, the electric field is described by

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\,\hat{r} \tag{1.5}$$

where σ is the surface charge density, since – as shown earlier – free charge in a conductor is located on the surface.

1.4 Charged Spheres

Here we have a conducting, a homogeneously charged and an in-homogeneously charged sphere. Their total charge is Q. Finding the electric field for each case in- and outside the sphere is an exercise in using Gauss' law

$$\oint \mathbf{E} \cdot d\mathbf{S} = \frac{q}{\epsilon_0} \tag{1.6}$$

For all cases:

- Problem 1.1c showed that the electric field is directed radially outward from the center of the spheres.
- For $r > a$, $\mathbf{E}$ behaves as if caused by a point charge of magnitude of the total charge Q of the sphere, at the origin.

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\,\hat{\mathbf{r}}$$

As we have seen earlier, for a solid spherical conductor the electric field inside is zero (see figure 1.3). For a sphere with a homogeneous charge distribution the electric field at points inside the sphere increases with r. As the surface S increases, the amount of charge surrounded increases (see equation 1.6):

$$\mathbf{E} = \frac{Qr}{4\pi\epsilon_0 a^3}\,\hat{\mathbf{r}} \tag{1.7}$$

For points inside a sphere with an inhomogeneous charge distribution, we use Gauss' law (once again)

$$\mathbf{E}4\pi r^2 = 1/\epsilon_0 \int_0^{2\pi} \int_0^{\pi} \int_0^{r} \rho(\mathbf{r}') r'^2 \sin\theta' dr' d\phi' d\theta' \tag{1.8}$$

Implementing the volume charge distribution

$$\rho(r') = \rho_0 r'^n,$$

the integration over r' for $n > -3$ is straightforward:

$$\mathbf{E} = \frac{\rho_0 r^{n+1}}{\epsilon_0 (n+3)}\,\hat{\mathbf{r}}, \tag{1.9}$$

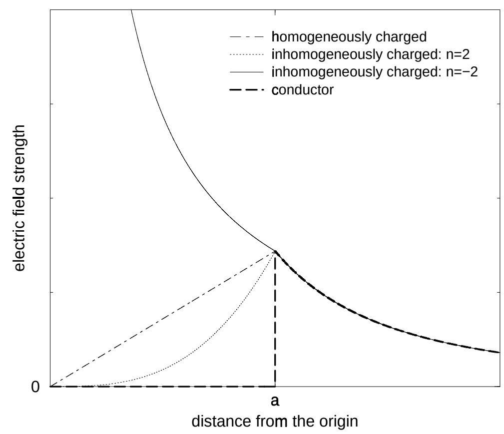

Figure 1.3: *Electric field for differently charged spheres of radius a. The electric field outside the spheres is the same for all, since the total charge is Q in all cases.*

where

$$\begin{aligned} Q &= \int_0^a 4\pi\rho_0 r^{n+2} dr = \frac{4\pi\rho_0 a^{n+3}}{n+3} \quad \Leftrightarrow \\ \rho_0 &= \frac{Q(n+3)}{4\pi a^{n+3}} \end{aligned} \tag{1.10}$$

It can easily be verified that for $n = 0$, we have the case of the homogeneously charged sphere (equation 1.7). The electric field as a function of distance are plotted in figure 1.3 for the conductor, the homogeneously charged sphere and in-homogeneously charged spheres with $n = -2, 2$.

1.5 Charge Density for a Hydrogen Atom

The potential of a neutral hydrogen atom is

$$\Phi(r) = \frac{q}{4\pi\epsilon_0}\frac{e^{-\alpha r}}{r}\left(1 + \frac{\alpha r}{2}\right) \tag{1.11}$$

where α equals 2 divided by the Bohr radius. If we calculate the Laplacian, we obtain the volume charge density ρ, through Poisson's equation

$$\frac{\rho}{\epsilon_0} = \nabla^2\Phi$$

Using the Laplacian for spherical coordinates (see back-cover of Jackson), the result for $r > 0$ is

$$\rho(r) = -\frac{\alpha^3 q}{8\pi r^2} e^{-\alpha r} \tag{1.12}$$

For the case of $r \to 0$

$$\lim_{r\to 0} \Phi(r) = \lim_{r\to 0} \frac{q}{4\pi\epsilon_0 r} \tag{1.13}$$

From section 1.7 in Jackson we have (J1.31):

$$\nabla^2(1/r) = -4\pi\delta(r) \tag{1.14}$$

Combining (1.13), (1.14) and Poisson's equation, we get for $r \to 0$

$$\rho(r) = q\delta(r) \tag{1.15}$$

We can multiply the right side of equation (1.15) by $e^{-\alpha r}$ without consequences. This allows for a more elegant way of writing the discrete and the continuous parts together

$$\rho(r) = \left(\delta(r) - \frac{\alpha^3}{8\pi r^2}\right) q e^{-\alpha r} \tag{1.16}$$

The discrete part represents the stationary proton with charge q. Around the proton orbits an electron with charge $-q$. The continuous part of the charge density function is more a statistical distribution of the location of the electron.

1.7 Charged Cylindrical Conductors

Two very long cylindrical conductors, separated by a distance d, form a capacitor. Cylinder 1 has surface charge density λ and radius a_1, and number 2 has surface charge density $-\lambda$ and radius a_2 (see figure 1.4). The electric field for each of the cylinders is radially directed outward

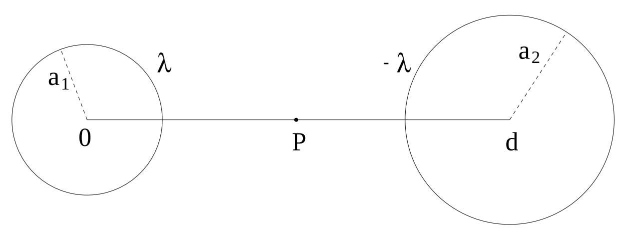

Figure 1.4: *Top view of two cylindrical conductors. The point P is located in the plane connecting the axes of the cylinders.*

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 |\mathbf{r}|}\,\hat{\mathbf{r}}, \tag{1.17}$$

where $\hat{\mathbf{r}}$ is the radially directed outward unit vector. Taking a point P on the plane connecting the axes of the cylinders, the electric field is constructed by superposition:

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0}\left(\frac{1}{r} + \frac{1}{d-r}\right) \tag{1.18}$$

The potential difference between the two cylinders is

$$\begin{aligned} |V_{a_2} - V_{a_1}| &= \frac{\lambda}{2\pi\epsilon_0}\int_{d-a_2}^{a_1}\left(\frac{1}{r} + \frac{1}{d-r}\right)dr \\ &= \frac{\lambda}{2\pi\epsilon_0}\left[\ln r + \ln(d-r)\right]_{d-a_2}^{a_1} \\ &= \frac{\lambda}{2\pi\epsilon_0}\left[\ln a_1 + \ln(d-a_1) + \ln(d-a_2) - \ln a_2\right] \end{aligned} \tag{1.19}$$

If we average the radii of the cylinders to $a_1 = a_2 = a$ and assume $d \gg a$, then the potential difference is

$$|V_{a_2} - V_{a_1}| \approx \frac{\lambda}{\pi\epsilon_0}\left(\ln\frac{d}{a}\right) \tag{1.20}$$

The capacitance per unit length of the system of cylinders is given by

$$C = \frac{\lambda}{|V_{a_1} - V_{a_2}|} \approx \frac{\pi\epsilon_0}{\ln(d/a)} \tag{1.21}$$

From here, we can obtain the diameter δ of wire necessary to have a certain capacitance C at a distance d:

$$\delta = 2a \approx 2d \cdot e^{-\frac{\pi\epsilon_0}{C}}, \tag{1.22}$$

where the permittivity in free space ϵ_0 is $8.854 \cdot 10^{-12} F/m$. If $C = 1.2 \cdot 10^{-11} F/m$ and

- $d = 0.5$ cm, the diameter of the wire is 0.1 cm.
- $d = 1.5$ cm, the diameter of the wire is 0.3 cm.
- $d = 5$ cm, the diameter of the wire is 1 cm.

1.13 Green's Reciprocity Theorem

Two infinite grounded parallel conducting plates are separated by a distance d. What is the induced charge on the plates if there is a point charge q in between the plates?

Split the problem up in two cases with the same geometry. The first is the situation as sketched by Jackson; two infinitely large grounded conducting plates, one at $x = 0$ and one at $x = d$ (see the right side of figure 1.5). In between the plates there is a point charge q at $x = x_0$. We will apply Green's reciprocity theorem using a "mirror" set-up. In this geometry there is no point charge but the plates have a fixed potential ψ_1 and ψ_2, respectively (see the left side of figure 1.5).

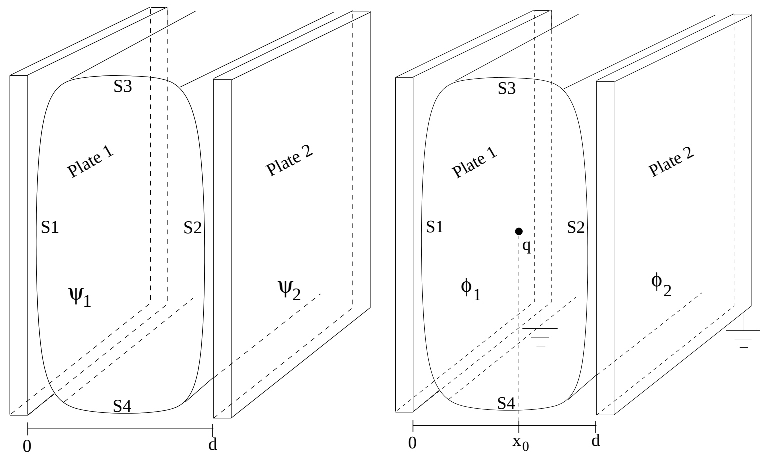

Figure 1.5: *Geometry of two conducting plates and a point-charge. S is the surface bounding the volume between the plates. The right picture is the situation of the imposed problem with the point charge between two grounded plates. The left side is a problem with the same plate geometry, but we know the potential ϕ on the plates.*

Green's theorem (J1.35) states

$$\int_{\mathbf{V}} \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) d\mathbf{V} = \oint_{\mathbf{S}} \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) d\mathbf{S} \tag{1.23}$$

The volume **V** is the space between the plates bounded by the surface **S**. **S1** and **S2** bound the plates and **S3** and **S4** run from plate 1 to plate 2 at + and - ∞, respectively. The normal derivative $\frac{\partial}{\partial n}$ at the surface **S** is directed outward from inside the volume **V**.

When the plates are grounded, the potential in the plates is zero. The potential is continuous across the boundary, so on $S1$ and $S2$ $\phi = 0$. Note that if the potential was not continuous the electric field ($\mathbf{E} = -\nabla\phi$) would go to infinity. At infinite distance from the point source, the potential is also zero:

$$\oint_{\mathbf{S}} \phi\frac{\partial\psi}{\partial n} d\mathbf{S} = 0 \tag{1.24}$$

The remaining part of the surface integral can be modified according to Jackson, page 36:

$$\frac{\partial\phi}{\partial n} = \nabla\phi\cdot\mathbf{n} = -\mathbf{E}\cdot\mathbf{n} \tag{1.25}$$

The electric field across a boundary with surface charge density σ is (Jackson, equation 1.22):

$$\mathbf{n}\cdot(\mathbf{E}_{conductor} - \mathbf{E}_{void}) = \sigma/\epsilon_0 \tag{1.26}$$

However, inside the conductor $\mathbf{E} = 0$, therefore

$$\mathbf{n}\cdot\mathbf{E}_{void} = -\sigma/\epsilon_0, \tag{1.27}$$

for each of the plates. The total surface integral in equation 1.23 is then

$$\oint_{\mathbf{S}} \psi \frac{\partial \phi}{\partial n} d\mathbf{S} = \int_{\mathbf{S1}} \psi_1 \frac{\sigma_1}{\epsilon_0} d\mathbf{S} + \int_{\mathbf{S2}} \psi_2 \frac{\sigma_2}{\epsilon_0} d\mathbf{S} \tag{1.28}$$

In case of the plates of fixed potential ψ, the legs **S3** and **S4** have opposite potential and thus cancel. Using

$$\int_{\mathbf{S}} \sigma d\mathbf{S} = Q, \tag{1.29}$$

the surface integral of Greens theorem is

$$\oint_{\mathbf{S}} \psi \frac{\partial \phi}{\partial n} d\mathbf{S} = 1/\epsilon_0 (\psi_1 Q_{\mathbf{S1}} + \psi_2 Q_{\mathbf{S2}}) \tag{1.30}$$

In the volume integral in equation (1.23), the mirror case of the charged boundaries includes no free charges:

$$\nabla^2 \psi = -(\text{total charge})/\epsilon_0 = 0 \tag{1.31}$$

Applying Gauss' law to the case with the point charge gives us

$$\nabla^2 \phi = -(\text{total charge})/\epsilon_0 = -q/\epsilon_0 \delta(x - x_0), \tag{1.32}$$

where x_0 is the x-coordinate of the point source location. The volume integral will be

$$\int_{\mathbf{V}} -q/\epsilon_0 \delta(x - x_0) \psi(x) d\mathbf{V} = -q/\epsilon_0 \psi(x_0) \tag{1.33}$$

For two plates with fixed potentials, the potential in between is a linear function

$$\psi(x_0) = \psi_1 + (\frac{\psi_2 - \psi_1}{d})(1 - x_0)d = x_0\psi_1 + \psi_2(1 - x_0) \tag{1.34}$$

Green's theorem is now reduced to equation (1.30) and equation (1.34) in (1.33):

$$-q(x_0\psi_1 + (1 - x_0)\psi_2) = Q_{S1}\psi_1 + Q_{S2}\psi_2 \tag{1.35}$$

Since this equality must hold for all potentials, the charges on the plates must be

$$Q_{S1} = -qx_0 \text{ and } Q_{S2} = -q(1 - x_0) \tag{1.36}$$

Chapter 2

Boundary-Value Problems in Electrostatics: 1

2.2 The Method of Image Charges

a. The Potential Inside the Sphere

This problem is similar to the example shown on pages 58, 59 and 60 of Jackson. The electric field due to a point charge q inside a grounded conducting spherical shell can also be created by the point charge and an image charge q' only. For reasons of symmetry it is evident that q' is located on the line connecting the origin and q. The goal is to find the location and the magnitude of the image charge. The electric field can then be described by superposition of point charges:

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\epsilon_0|\mathbf{x}-\mathbf{y}|} + \frac{q'}{4\pi\epsilon_0|\mathbf{x}-\mathbf{y}'|} \tag{2.1}$$

In figure 2.1 you can see that $\mathbf{x}$ is the vector connecting origin and observation point. $\mathbf{y}$ connects the origin and the unit charge q. Finally, $\mathbf{y}'$ is the connection between the origin and the image charge q'. Next, we write the vectors in terms of a scalar times their unit vector and factor the scalars y' and x out of the denominators:

$$\Phi(\mathbf{x}) = \frac{q/4\pi\epsilon_0}{x|\mathbf{n}-\frac{y}{x}\mathbf{n}'|} + \frac{q'/4\pi\epsilon_0}{y'|\mathbf{n}'-\frac{x}{y'}\mathbf{n}|} \tag{2.2}$$

The potential for $x = a$ is zero, for all possible combinations of $\mathbf{n}\cdot\mathbf{n}'$. The magnitude of the image charge is

$$q' = -\frac{a}{y}q \tag{2.3}$$

at distance

$$y' = \frac{a^2}{y} \tag{2.4}$$

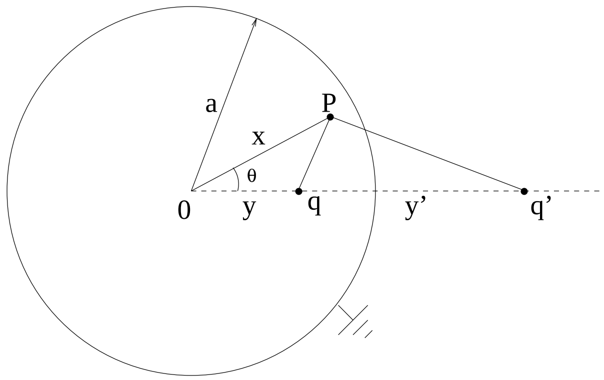

Figure 2.1: *A point charge q in a grounded spherical conductor. q' is the image charge.*

This is the same result as for the image charge inside the sphere and the point charge outside (like in the Jackson example). After implementing the amount of charge (2.3) and the location of the image (2.4) in (2.1), potential in polar coordinates is

$$\Phi(r,\theta) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{y^2+r^2-2yr\cos\theta}} - \frac{\left(\frac{a}{y}\right)}{\sqrt{\left(\frac{a^2}{y}\right)^2 + r^2 - 2\left(\frac{a^2}{y}\right) r\cos\theta}}\right), \tag{2.5}$$

where θ is the angle between the line connecting the origin and the charges and the line connecting the origin and the point P (see figure 2.1). r is the length of the vector connecting the origin and observation point P.

b. The Induced Surface Charge Density

The surface charge density on the sphere is

$$\sigma = \epsilon_0 \left(\frac{\partial\Phi}{\partial r}\right)_{r=a} \tag{2.6}$$

Differentiating equation (2.5) is left to the reader, but the result is

$$\sigma = -\frac{q}{4\pi a^2}\left(\frac{a}{y}\right)\frac{1-\left(\frac{a}{y}\right)^2}{\left(1+\left(\frac{a}{y}\right)^2 - 2\frac{a}{y}\cos\theta\right)^{3/2}} \tag{2.7}$$

c. The Force on the Point Charge q

The force on the point charge q by the field of the induced charges on the conductor is equal to the force on q due to the field of the image charge:

$$\mathbf{F} = q\mathbf{E}' \tag{2.8}$$

The electric field at y due to the image charge at y' is directed towards the origin and of magnitude

$$|\mathbf{E}'| = \frac{q'}{4\pi\epsilon_0 (y'-y)^2} \tag{2.9}$$

We already computed the values for y' and q' in equation (2.4) and (2.3), respectively. The force is also directed towards the origin of magnitude

$$|\mathbf{F}| = \frac{1}{4\pi\epsilon_0}\frac{q^2}{a^2}\left(\frac{a}{y}\right)\left(1-\left(\frac{a}{y}\right)^2\right)^{-2} \tag{2.10}$$

d. What If the Conductor Is Charged?

Keeping the sphere at a fixed non-zero potential requires net charge on the conducting shell. This can be imaged as an extra image charge at the center of the spherical shell. If we now compute the force on the conductor by means of the images, the result *will* differ from section c.

2.7 An Exercise in Green's Theorem

a. The Green Function

The Green function for a half-space ($z > 0$) with Dirichlet boundary conditions can be found by the method of images. The potential field of a point source of unit magnitude at z' from an infinitely large grounded plate in the x-y plane can be replaced by an image geometry with the unit charge q and an additional (image) charge at $z = -z'$ of magnitude $q' = -q$. The situation is sketched in figure 2.2. The potential due to the two charges is the Green function G_D:

$$\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}} \tag{2.11}$$

b. The Potential

The Green function as defined in equation (2.11) can serve as the "mirror set-up" required in Green's theorem:

$$\int_{\mathbf{V}} \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) d\mathbf{V} = \oint_{\mathbf{S}} \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) d\mathbf{S} \tag{2.12}$$

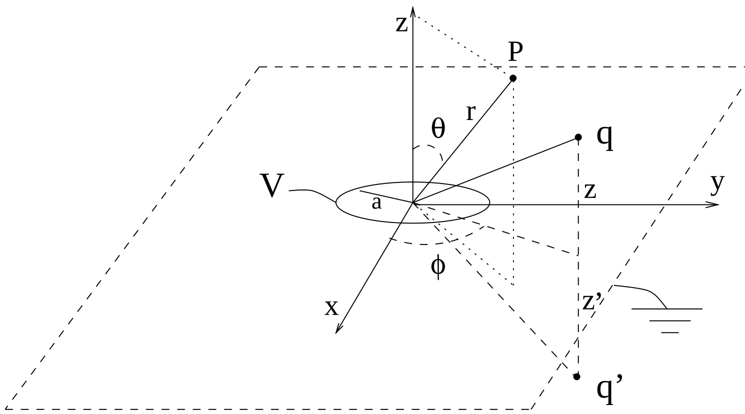

Figure 2.2: A very large grounded surface in which a circular shape is cut out and replaced by a conducting material of potential V.

with $G_D = \psi$ and $\Phi = \phi$. There are no charges the volume $z > 0$, so Laplace equation holds throughout the half-space $\mathbf{V}$:

$$\nabla^2 \Phi = 0 \tag{2.13}$$

Chapter one in Jackson (J1.39) showed

$$\nabla^2 G_D = -4\pi\delta(\rho - \rho') \tag{2.14}$$

leaving $\Phi(\rho')$ after performing the volume integration. The Green function on the surface S (G_D) is constructed with the assumption that part of the surface (the base, if you will) is grounded, and the other parts stretch to infinity. Therefore

$$\oint_{\mathbf{S}} G_D d\mathbf{S} = 0 \tag{2.15}$$

The potential $\Phi = V$ in the circular area with radius a, but everywhere else $\Phi = 0$. Also:

$$\oint_{\mathbf{S}} \nabla G_D \cdot \hat{\mathbf{n}} d\mathbf{S} = -\oint_{\mathbf{S}} \frac{\partial G_D}{\partial z} d\mathbf{S}, \tag{2.16}$$

since the normal $\hat{\mathbf{n}}$ is in the negative z-direction $-\hat{\mathbf{k}}$. Thus we are left with the following remaining terms in Green's theorem (in cylindrical coordinates):

$$\Phi(\mathbf{r}') = \epsilon_0 V \int_0^a \int_0^{2\pi} \frac{\partial G_D}{\partial z} \rho d\rho d\phi \tag{2.17}$$

From here on we will exchange the primed and unprimed coordinates. This is OK, since the reciprocity theorem applies. Some algebra left to the reader leads to

$$\Phi(\mathbf{r}) = \frac{Vz}{2\pi} \int_0^a \int_0^{2\pi} \frac{\rho' d\rho' d\phi'}{(\rho^2 + \rho'^2 + z^2 - 2\rho\rho' \cos(\phi - \phi'))^{3/2}} \tag{2.18}$$

c. The Potential on the z-axis

For $\rho = 0$, general solution (2.18) simplifies to

$$\begin{aligned} \Phi(\rho, \phi) &= \frac{Vz}{2\pi} \int_0^a \int_0^{2\pi} \frac{\rho' d\rho' d\phi'}{(\rho'^2 + z^2)^{3/2}} \\ &= -Vz \left[\frac{1}{\sqrt{\rho'^2 + z^2}} \right]_0^a \\ &= V \left(1 - \frac{z}{\sqrt{a^2 + z^2}} \right) \end{aligned} \tag{2.19}$$

d. An Approximation

Slightly rewriting equation (2.18):

$$\Phi(\mathbf{r}) = \frac{Vz}{2\pi\left(\rho^2 + z^2\right)^{3/2}} \int_0^a \int_0^{2\pi} \frac{\rho' d\rho' d\phi'}{\left(1 + \frac{\rho'^2 - 2\rho\rho' \cos(\phi - \phi')}{\rho^2 + z^2}\right)^{3/2}} \tag{2.20}$$

The denominator in the integral can be approximated by a binomial expansion. The first three terms of the approximation give

$$\Phi(\mathbf{r}) \approx \frac{Va^2 z}{2\left(\rho^2 + z^2\right)^{3/2}} \left(1 - \frac{3a^2}{4\left(\rho^2 + z^2\right)} + \frac{5a^4}{8\left(\rho^2 + z^2\right)^2} + \frac{15a^2\rho^2}{8\left(\rho^2 + z^2\right)^2} \right) \tag{2.21}$$

Along the axis ($\rho = 0$) the expression simplifies to

$$\Phi(\phi, z) \approx \frac{Va^2}{2z^2} \left(1 - \frac{3a^2}{4z^2} + \frac{5a^4}{8z^4} \right) \tag{2.22}$$

This is the same result when we expand expression (2.19):

$$\begin{aligned} \Phi(\rho, \phi) &= V \left(1 - \frac{z}{\sqrt{a^2 + z^2}} \right) \\ &= V \left(1 - \left(1 + \frac{a^2}{z^2} \right)^{-1/2} \right) \\ &\approx \frac{Va^2}{2z^2} \left(1 - \frac{3a^2}{4z^2} + \frac{5a^4}{8z^4} \right) \end{aligned} \tag{2.23}$$

2.9 Two Halves of a Conducting Spherical Shell

A conducting spherical shell consists of two halves. The cut plane is perpendicular to the homogeneous field (see figure 2.3). The goal is to investigate the force between the two halves introduced

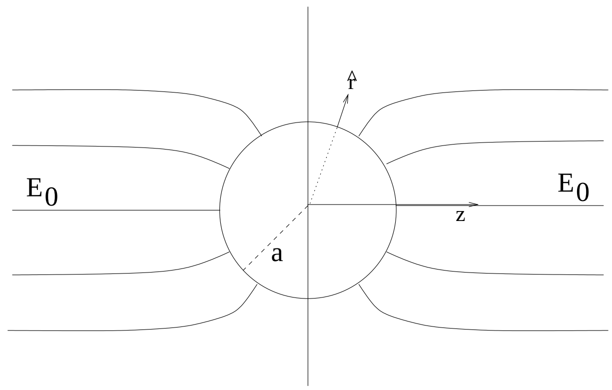

Figure 2.3: A spherical conducting shell in a homogeneous electric field directed in the z-direction.

by the induced charges.

The electric field due to the induced charges on the shell is (see [2], p. 51):

$$\mathbf{E}_{ind} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{r}} \tag{2.24}$$

You can see this as the resulting field in a capacitor with one of the plates at infinity. The electric field inside the conducting shell is zero. Therefore the external field has to be of the same magnitude (see figure 2.4).

The force of the external field on an elementary surface dS of the conductor is:

$$d\mathbf{F} = \mathbf{E}_{ext}dq = \frac{\sigma^2 d\mathbf{S}}{2\epsilon_0}, \tag{2.25}$$

directed radially outward from the sphere's center. From the symmetry we can see that all forces cancel, except the component in the direction of the external field.[1]

a. An Uncharged Shell

The derivation of the induced charge density on a conducting spherical shell in a homogeneous electrical field E_0 is given ([3], p. 64). The homogeneous field is portrayed by point charges of opposite magnitude at + and − infinity. Next, the location and magnitude of the image charges are computed. The result is

$$\sigma(\theta) = 3\epsilon_0 E_0 \cos\theta \tag{2.26}$$

[1]The external field is a superposition of the homogeneous electric field **plus** the electric field due to the induced charges excluding dq! This external field is perpendicular to the surface of the conductor with magnitude $\sigma/(2\epsilon_0)$. This is not addressed in Jackson.

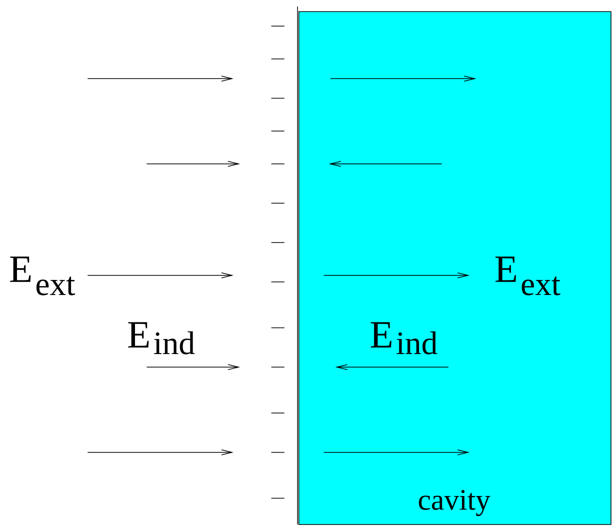

Figure 2.4: Zooming in on that small part of the conductor with induced charges, where the external field is at normal incidence.

When we plug this result into equation (2.25), we get for the horizontal component of the force ($d\mathbf{F}_z$) on an elementary surface:

$$d\mathbf{F}_z = d\mathbf{F}\cos\theta = \frac{9}{2}\epsilon_0 E_0^2 \cos^3\theta d\mathbf{S}. \tag{2.27}$$

Now we can integrate to get the total force on the sphere halves. From symmetry we can also see that the force on the left half is opposite of that on the right half (see figure 2.3). So we integrate over the right half and multiply by two to get the total net force:

$$\begin{aligned}
\mathbf{F_z} &= 2\int_0^{2\pi}\int_0^{\pi/2} \frac{9}{2}\epsilon_0 E_0^2 \cos^3\theta a^2 \sin\theta d\theta d\phi \ \hat{\mathbf{z}} \\
&= 9\pi a^2 \epsilon_0 E_0^2 \int_0^{\pi/2} \cos^3\theta \sin\theta d\theta \ \hat{\mathbf{z}} \\
&= 9\pi a^2 \epsilon_0 E_0^2 \left[-1/4\cos^4\theta\right]_0^{\pi/2} \ \hat{\mathbf{z}} \\
&= \frac{9}{4}\pi a^2 \epsilon_0 E_0^2 \hat{\mathbf{z}}
\end{aligned} \tag{2.28}$$

b. A Shell with Total Charge Q

When the shell has a total charge Q it changes the charge density of equation (2.26) to

$$\sigma(\theta) = 3\epsilon_0 E_0 \cos\theta + \frac{Q}{4\pi a^2} \tag{2.29}$$

When we plug this expression into equation (2.25) and compute again the net (horizontal) component of the force, we find that

$$\mathbf{F_z} = \left(\frac{9}{4}\pi a^2 \epsilon_0 E_0^2 + \frac{Q^2}{32\pi\epsilon_0 a^2} + \frac{E_0 Q}{2}\right)\hat{\mathbf{z}} \tag{2.30}$$

The total force is bigger then for the uncharged case. This makes sense when we look at equation (2.25); when the shell is charged there is more charge per unit volume to, hence the force is bigger.

2.10 A Conducting Plate with a Boss

a. σ On the Boss

By inspection it can be seen that the system of images as proposed in figure (2.5) fits the geometry and the boundary conditions of our problem. We can write the potential as a function of these four point charges. This is done in Jackson (p. 63). It has to be noted that R has to be chosen at infinity to apply to the homogeneous character of the field. The potential can then be described by expanding "the radicals after factoring out the R^2."

$$\begin{aligned}\Phi(r,\theta) &= \frac{Q}{4\pi\epsilon_0}\left(-\frac{2}{R^2}r\cos\theta + \frac{2a^3}{R^2 r^2}\cos\theta\right) + \ldots \\ &= -E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta \end{aligned} \tag{2.31}$$

The surface charge density on the boss ($r = a$) is

$$\sigma = -\,\epsilon_0 \left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = 3\epsilon_0 E_0 \cos\theta \tag{2.32}$$

b. The Total Charge on the Boss

The total charge Q on the boss is merely an integration over half a sphere with radius a:

$$\begin{aligned} Q &= 3\epsilon_0 E_0 \int_0^{2\pi}\int_0^{\pi/2} \cos\theta a^2 \sin\theta d\theta d\phi \\ &= 3\epsilon_0 E_0 2\pi a^2 \left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2} \\ &= 3\epsilon_0 E_0 \pi a^2 \end{aligned} \tag{2.33}$$

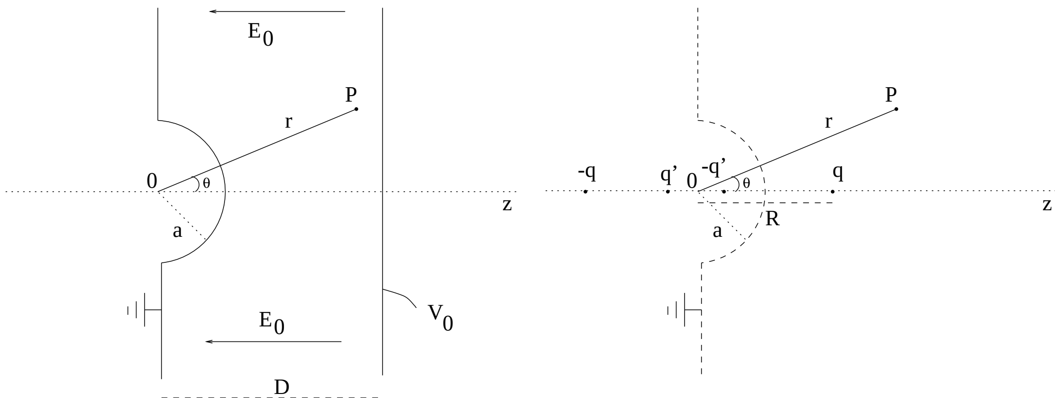

Figure 2.5: *On the left is the geometry of the problem: two conducting plates separated by a distance D. One of the plates has a hemispheric boss of radius a. The electric field between the two plates is E_0. On the right is the set of charges that image the field due to the conducting plates. In part a and b, $R \to \infty$ to image a homogeneous field. In c, $R = d$.*

c. The Charge On the Boss Due To a Point Charge

Now we do not have a homogeneous field to image, but the result of a point charge on a grounded conducting plate with the boss. Again we use the method of images to replace the system with the plate by one entirely consisting of point charges. Checking the boundary conditions leads to the same set of four charges as drawn in figure 2.5. The only difference is that R is not chosen at infinity to mimic the homogeneous field, but $R = d$. The potential is the superposition of the point charge q at distance d and its three image charges:

$$\Phi(r,\theta) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{(r^2+d^2+2rd\cos\theta)^{1/2}} - \frac{1}{(r^2+d^2-2rd\cos\theta)^{1/2}} \right.$$
$$\left. - \frac{a}{d\left(r^2+\frac{a^4}{d^2}+\frac{2a^2r}{d}\cos\theta\right)^{1/2}} + \frac{a}{d\left(r^2+\frac{a^4}{d^2}-\frac{2a^2r}{d}\cos\theta\right)^{1/2}}\right) \tag{2.34}$$

The charge density on the boss is

$$\sigma = -\epsilon_0 \left.\frac{\partial\Phi}{\partial r}\right|_{r=a} \tag{2.35}$$

The total amount of charge is the surface charge density integrated over the surface of the boss:

$$Q = 2\pi a^2 \int_0^{\pi/2} \sigma \sin\theta d\theta \tag{2.36}$$

The differentiation of equation (2.34) to obtain the the surface charge density and the following integration in equation (2.36) are left to the reader. The resulting total charge is[2]

$$Q = -q\left[1 - \frac{d^2 - a^2}{d\sqrt{d^2 + a^2}}\right] \tag{2.37}$$

2.11 Line Charges and the Method of Images

a. Magnitude and Position of the Image Charge(s)

Analog to the situation of point charges in previous image problems, one image charge of opposite magnitude at distance $\frac{b^2}{R}$ (see figure 2.6) satisfies the conditions the boundary conditions

$$\begin{aligned} \lim_{r\to\infty} \Phi(r,\phi) &= 0 \quad \text{and} \\ \Phi(b,\phi) &= V_0 \end{aligned}$$

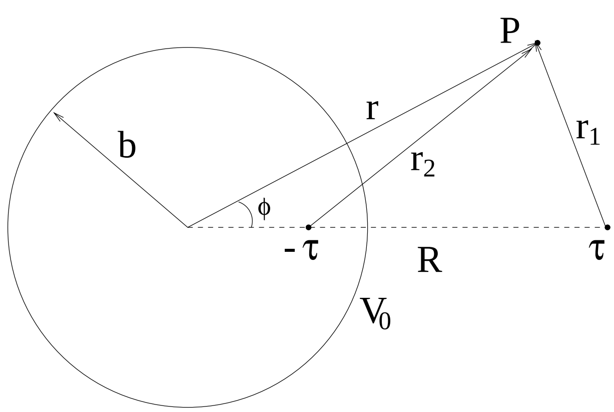

Figure 2.6: *A cross sectional view of a long cylinder at potential V_0 and a line charge τ at distance R, parallel to the axis of the cylinder. The image line charge $-\tau$ is placed at b^2/R from the axis of the cylinder to realize a constant potential V_0 at radius $r = b$.*

[2] I have chosen to keep Q as the symbol for the total charge. Jackson calls it q'. I find this confusing since the primed q has been used for the image of q.

b. The Potential

The potential in polar coordinates is simply a superposition of the line charge τ and the image line charge τ' with the conditions as proposed in section a. The result is

$$\Phi(r,\phi) = \frac{\tau}{4\pi\epsilon_0}\ln\left(\frac{(R^2r^2+b^4-2rRb^2\cos\phi)}{R^2(r^2+R^2-2Rr\cos\phi)}\right) \tag{2.38}$$

For the far field case ($r >> R$) we can factor out $(Rr)^2$. The b^4 and R^2 in equation (2.38) can be neglected:

$$\Phi(r,\phi) \approx \frac{\tau}{4\pi\epsilon_0}\ln\left(\frac{(Rr)^2(1-\frac{2b^2}{Rr}\cos\phi)}{(Rr)^2(1-\frac{2R}{r}\cos\phi)}\right) \tag{2.39}$$

The first order Taylor expansion is

$$\begin{aligned}\Phi(r,\phi) &\approx \frac{\tau}{4\pi\epsilon_0}\ln\left((1-\frac{2b^2}{Rr}\cos\phi)(1+\frac{2R}{r}\cos\phi)\right) \\ &\approx \frac{\tau}{4\pi\epsilon_0}\ln\left((1+\left(\frac{2R}{r}\cos\phi-\frac{2b^2}{Rr}\cos\phi\right)\right) \\ &\approx \frac{\tau}{2\pi\epsilon_0}\frac{(R^2-b^2)}{Rr}\cos\phi \qquad \text{(using } \ln(1+x)\approx x) \end{aligned} \tag{2.40}$$

c. The Induced Surface Charge Density

$$\sigma(\phi) = -\epsilon_0\left.\frac{\partial\Phi}{\partial r}\right|_{r=b} \tag{2.41}$$

Differentiation of equation (2.38) and substituting $r = b$:

$$\begin{aligned}\sigma(\phi) &= -\frac{\tau}{4\pi}\left(\frac{2bR^2-2Rb^2\cos\phi}{R^2b^2+R^2+b^4-2b^3R\cos\phi}-\frac{R^2(2b-2R\cos\phi}{R^2(b^2+R^2-2Rb\cos\phi}\right) \\ &= -\frac{\tau}{2\pi}\left(\frac{(R/b)^2-1}{(R/b)^2+1-2(R/b)\cos\phi}\right)\end{aligned} \tag{2.42}$$

When $R/b = 2$, the induced charge as a function of ϕ is

$$\sigma(\phi)|_{R=2b} = -\frac{\tau}{2\pi b}\left(\frac{3}{5-4\cos\phi}\right) \tag{2.43}$$

When the position of the line charge τ is four radii from the center of the cylinder, the surface charge density is

$$\sigma(\phi)|_{R=4b} = -\frac{\tau}{2\pi b}\left(\frac{15}{17-8\cos\phi}\right) \tag{2.44}$$

The graphs for either case are drawn in figure 2.7.

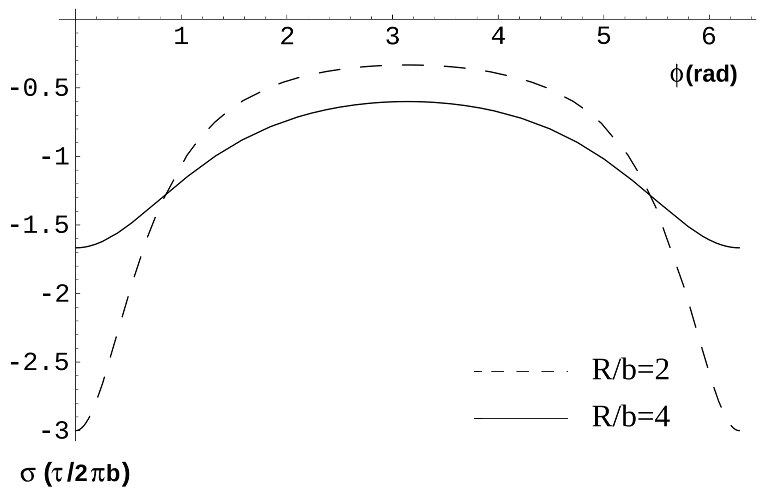

Figure 2.7: The behavior of the surface charge density σ with angle for two different ratios between the radius of the cylinder and the distance to the line charge.

d. The Force on the Line Charge

The force per meter on the line charge is Coulomb's law:

$$\mathbf{F} = \tau \mathbf{E}(R, 0) = -\frac{\tau^2}{2\pi\epsilon_0} \frac{1}{(R^2 - b^2)} \hat{\imath} \quad \text{per meter} \tag{2.45}$$

where $\hat{\imath}$ is the directed from the the axis of the cylinder to the line charge, perpendicular to the line charge and the cylinder axis.

2.13 Two Cylinder Halves at Constant Potentials

a. The Potential inside the Cylinder

In this case (see figure 2.8) there are no free charges in the area of interest. Therefore the potential Φ inside the cylinder obeys Laplace's equation:

$$\nabla^2 \Phi = 0 \tag{2.46}$$

We can write the potential in cylindrical coordinates and separate the variables:

$$\Phi(\rho, \phi) = R(r) F(\phi) \tag{2.47}$$

The general solution is (see J2.71):

$$\Phi(\rho,\phi) \quad = \quad a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty}\left[a_n\rho^n \sin(n\phi+\alpha_n) + b_n\rho^{-n}\cos(n\phi+\alpha_n)\right] \tag{2.48}$$

From this geometry it is obvious that at the center $\rho = 0$ the solution may not blow up, so:

$$b_n = b_0 = 0 \tag{2.49}$$

This results in a potential

$$\Phi(\rho,\phi) \quad = \quad a_0 + \sum_{n=1}^{\infty} a_n\rho^n \sin(n\phi+\alpha_n) \tag{2.50}$$

The next step is to implement the boundary conditions

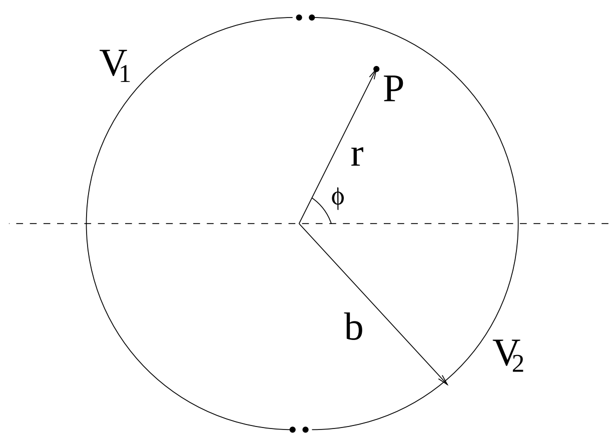

Figure 2.8: Cross-section of two cylinder halves with radius b at constant potentials V_1 and V_2.

$$\begin{aligned}\Phi(b,\phi) = V_2 = a_0 + \sum_{n=1}^{\infty} a_n b^n \sin(n\phi+\alpha_n) \quad \text{for } (-\pi/2 < \phi < \pi/2)\\ \Phi(b,\phi) = V_1 = a_0 + \sum_{n=1}^{\infty} a_n b^n \sin(n\phi+\alpha_n) \quad \text{for } (\pi/2 < \phi < 3\pi/2)\end{aligned} \tag{2.51}$$

b. The Surface Charge Density

$$\sigma = -\,\epsilon_0 \left.\frac{\partial\Phi}{\partial r}\right|_{r=b} \tag{2.52}$$

2.23 A Hollow Cubical Conductor

a. The Potential inside the Cube

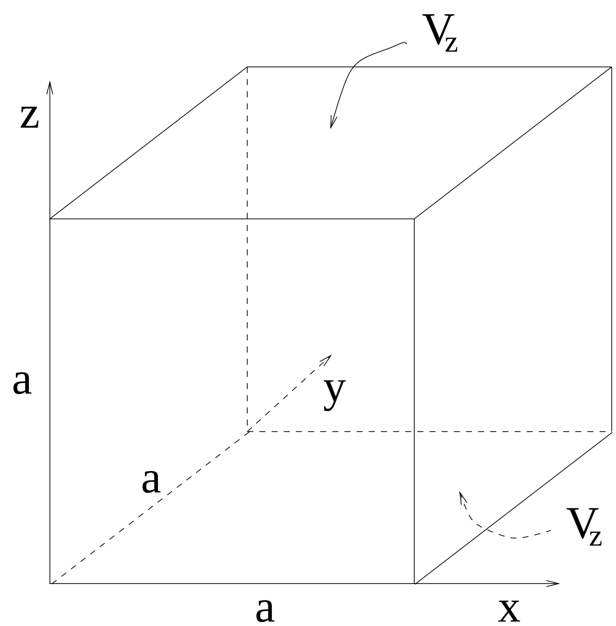

Figure 2.9: A hollow cube, with all sides but z=0 and z=a grounded.

$$\nabla^2\Phi = 0 \tag{2.53}$$

Separating the variables:

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0. \tag{2.54}$$

x and y can vary independently so each term must be equal to a constant $-\alpha^2$:

$$\frac{1}{X}\frac{d^2X}{dx^2} + \alpha^2 = 0 \quad \Rightarrow \quad X = A\cos\alpha x + B\sin\alpha x \tag{2.55}$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} + \beta^2 = 0 \quad \Rightarrow \quad Y = C\cos\beta y + D\sin\beta y \tag{2.56}$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} + \gamma^2 = 0 \quad \Rightarrow \quad Z = E\sinh(\gamma z) + F\cosh(\gamma z), \tag{2.57}$$

where $\gamma^2 = \alpha^2 + \beta^2$. The boundary conditions determine the constants:

$$\Phi(0, y, z) = 0 \quad \Rightarrow \quad A = 0$$
$$\Phi(a, y, z) = 0 \quad \Rightarrow \quad \alpha_n = n\pi/a \quad (n = 1, 2, 3, ...)$$

$$\begin{aligned}\Phi(x,0,z)=0 \quad &\Rightarrow \quad C=0\\ \Phi(x,a,z)=0 \quad &\Rightarrow \quad \beta_m = m\pi/a \quad (m=1,2,3,...)\\ &\Rightarrow \quad \gamma_{nm}=\pi\sqrt{n^2+m^2}\end{aligned}$$

The solution is thus reduced to

$$\Phi(x,y,z)=\sum_{n,m=1}^{\infty}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\left[A_{nm}\sinh\left(\frac{\gamma_{nm}z}{a}\right)+B_{nm}\cosh\left(\frac{\gamma_{nm}z}{a}\right)\right] \tag{2.58}$$

Now, let's use the last boundary conditions to find the coefficients A_nm and B_nm. The top and bottom of the cube are held at a constant potential V_z, so

$$\Phi(x,y,0)=V_z=\sum_{n,m=1}^{\infty}B_{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right) \tag{2.59}$$

This means that B_{nm} are merely the coefficients of a double Fourier series (see for instance [1] on Fourier series):

$$B_{nm}=\frac{4V_z}{a^2}\int_0^a\int_0^a\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)dxdy \tag{2.60}$$

It can be easily shown that the individual integrals in equation (2.60) are zero for even integer values and $\frac{2a}{n\pi}$ for n is odd. Thus B_nm is

$$B_{nm}=\frac{16V_z}{\pi^2 nm} \quad \text{for odd } (n,m) \tag{2.61}$$

The top of the cube is also at constant potential V_z, so

$$\begin{aligned}\Phi(x,y,0) &= V_z=\Phi(x,y,a)\Leftrightarrow\\ B_{nm} &= A_{nm}\sinh(\gamma_{nm})+B_{nm}\cosh(\gamma_{nm})\Leftrightarrow\\ A_{nm} &= B_{nm}\frac{1-\cosh(\gamma_{nm})}{\sinh(\gamma_{nm})}\end{aligned} \tag{2.62}$$

Substituting the expressions for A_{nm} and B_{nm} into equation (2.58), gives us

$$\begin{aligned}\Phi(x,y,z)=\frac{16V_z}{\pi^2}\sum_{n,m\ odd}^{\infty}\frac{1}{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)&\left[\frac{1-\cosh(\gamma_{nm})}{\sinh(\gamma_{nm})}\sinh\left(\frac{\gamma_{nm}z}{a}\right)\right.\\ &\left.+\cosh\left(\frac{\gamma_{nm}z}{a}\right)\right],\end{aligned} \tag{2.63}$$

where $\gamma_{nm}=\pi\sqrt{n^2+m^2}$.

b. The Potential at the Center of the Cube

The potential at the center of the cube is

$$\begin{aligned}\Phi(\frac{a}{2},\frac{a}{2},\frac{a}{2})=\frac{16V_z}{\pi^2}\sum_{n,m\ odd}^{\infty}\frac{1}{nm}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{m\pi}{2}\right)&\left[\frac{1-\cosh(\gamma_{nm})}{\sinh(\gamma_{nm})}\sinh\left(\frac{\gamma_{nm}}{2}\right)\right.\\ &\left.+\cosh\left(\frac{\gamma_{nm}}{2}\right)\right]\end{aligned} \tag{2.64}$$

With just $n, m = 1$, the potential at the center is

$$\frac{16V_z}{\pi^2}\left[\frac{1-\cosh(\sqrt{2}\pi)}{\sinh(\sqrt{2}\pi)}\sinh\left(\frac{\pi}{\sqrt{2}}\right)+\cosh\left(\frac{\pi}{\sqrt{2}}\right)\right]\approx 0.347546V_z \tag{2.65}$$

When we add the two terms $(n = 3,\ m = 1)$ and $(n = 1,\ m = 3)$, the potential is $0.332498V_z$.

c. The Surface Charge Density

The surface charge density on the top surface of the cube is given by

$$\sigma = -\epsilon_0 \left.\frac{\partial \Phi}{\partial z}\right|_{z=a} \tag{2.66}$$

In the appendix it is shown that the differentiation of the hyperbolic sine is the hyperbolic cosine. Furthermore

$$\frac{d\cosh(az)}{dz} = a\sinh(az) \tag{2.67}$$

Using this equality in differentiating the expression for the potential in equation (2.63), we get

$$\frac{\partial \Phi}{\partial z} = \frac{16V_z}{\pi^2}\sum_{n,m\ odd}^{\infty}\frac{\gamma_{nm}}{nma}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\left[\frac{1-\cosh(\gamma_{nm})}{\sinh(\gamma_{nm})}\cosh\left(\frac{\gamma_{nm}z}{a}\right) + \sinh\left(\frac{\gamma_{nm}z}{a}\right)\right], \tag{2.68}$$

where $\gamma_{nm} = \pi\sqrt{n^2+m^2}$. Now we evaluate this expression for $z = a$:

$$\sigma = -\epsilon_0 \left.\frac{\partial \Phi}{\partial z}\right|_{z=a} =$$

$$-\frac{16\epsilon_0 V_z}{\pi^2}\sum_{n,m\ odd}^{\infty}\frac{\gamma_{nm}}{nma}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\left[\frac{1-\cosh(\gamma_{nm})}{\sinh(\gamma_{nm})}\cosh(\gamma_{nm})+\sinh(\gamma_{nm})\right] =$$

$$-\frac{16\epsilon_0 V_z}{\pi^2}\sum_{n,m\ odd}^{\infty}\frac{\gamma_{nm}}{nma}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\left[(1-\cosh(\gamma_{nm}))\coth(\gamma_{nm})+\sinh(\gamma_{nm})\right] \tag{2.69}$$

Further simplification??

Chapter 3

Practice Problems

3.1 Angle between Two Coplanar Dipoles

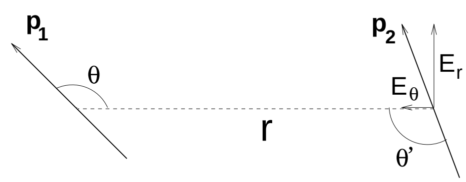

Figure 3.1: *Coplanar dipoles separated by a distance r.*

Two dipoles are separated by a distance r. Dipole $\mathbf{p_1}$ is fixed at an angle θ as defined in figure 3.1, while $\mathbf{p_2}$ is free to rotate. The orientation of the latter is defined by the angle θ' as defined in figure 3.1. Let us calculate the angular dependence between the two dipoles in equilibrium.

The electric field due to a dipole can be decomposed into a radial and a tangential component (D3.39):

$$\mathbf{E}(r,\theta) = \frac{2p\mathrm{cos}\theta}{4\pi\epsilon_0 r^3}\hat{\mathbf{r}} + \frac{p\mathrm{sin}\theta}{4\pi\epsilon_0 r^3}\hat{\boldsymbol{\theta}}, \tag{3.1}$$

where ϵ_0 is the dielectric permittivity. Writing $\mathbf{p_2}$ in terms of r and θ:

$$\mathbf{p}_2(r,\theta) = (\mathbf{p}_2 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} + (\mathbf{p}_2 \cdot \hat{\boldsymbol{\theta}} = p_2\mathrm{cos}\theta'\hat{\mathbf{r}} + p_2\mathrm{sin}\theta'\hat{\boldsymbol{\theta}} \tag{3.2}$$

The potential energy of the second dipole in the electric field due to the first dipole is

$$U_2(r,\theta,\theta') = -\mathbf{E}_1 \cdot \mathbf{p}_2 = -\frac{p_1 p_2}{4\pi\epsilon_0 r^3}(2\mathrm{cos}\theta\mathrm{cos}\theta' - \mathrm{sin}\theta\mathrm{sin}\theta') \tag{3.3}$$

The second dipole will rotate to minimize its potential energy, defining the angular dependence between the two dipoles:

$$\begin{aligned} \frac{\partial U_2}{\partial \theta'} &= 0 \quad \Leftrightarrow \\ \frac{p_1 p_2}{4\pi\epsilon_0 r^3}(2\cos\theta\sin\theta' + \sin\theta\cos\theta') &= 0 \quad \Leftrightarrow \\ 2\cos\theta\sin\theta' &= -\sin\theta\cos\theta' \quad \Leftrightarrow \\ \tan\theta' &= -(\tan\theta)/2. \end{aligned} \tag{3.4}$$

3.2 The Potential in Multipole Moments

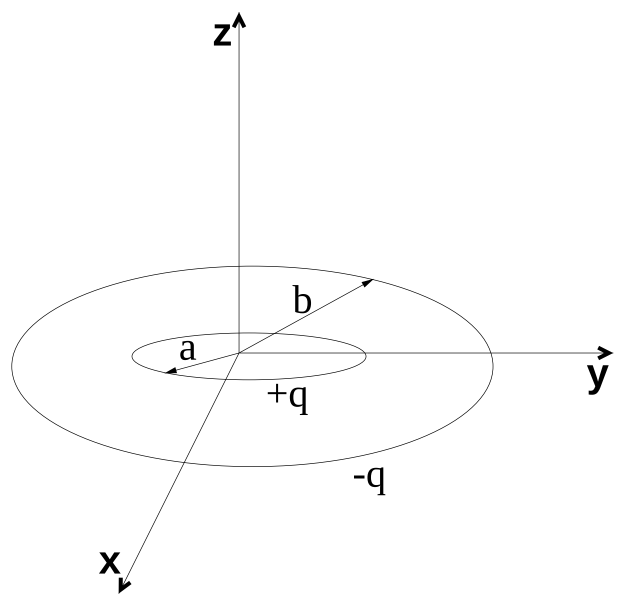

Figure 3.2: *Concentric rings of radii a and b. Their charge is q and $-q$, respectively.*

This exercise is how to find the potential due to two charged concentric rings (see figure 3.2) in terms of the monopole dipole and quadrupole moments. Discarding higher order moments (J4.10):

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 r} + \frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\epsilon_0 r^3} + \frac{1}{8\pi\epsilon_0 r^5}\sum_{i,j} x_i x_j Q_{ij} + ... \tag{3.5}$$

The monopole moment is zero since there is no net charge. The dipole moment is (J4.8):

$$\mathbf{p} = \int_{\mathbf{V}} \mathbf{r}\rho(\mathbf{r})d\mathbf{V}, \tag{3.6}$$

where the volume charge density for our case can be written in cylindrical coordinates:

$$\rho(\mathbf{r}) = \frac{q}{2\pi a}\delta(r-a)\delta(z) - \frac{q}{2\pi b}\delta(r-b)\delta(z) \tag{3.7}$$

After the x-component of the dipole moment is also written in cylindrical coordinates, the integral can be easily evaluated:

$$p_x = \int_{\mathbf{V}} x\rho(r,z)d\mathbf{V} = \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty} \cos\phi\rho(r,z)r^2 dr d\phi dz = 0, \tag{3.8}$$

because the $\cos\phi$ integrated over one period is zero. The y-component is zero, because the integration involves a sinusoid over one period. Finally, the z-component is zero, because

$$p_z \propto \int z\delta(z)dz = 0 \tag{3.9}$$

if the value 0 is within the integration limits.

The quadrupole moment is a tensor:

$$Q_{ij} = \int_{\mathbf{V}} (3x_i x_j - \mathbf{r}^2)\rho(\mathbf{r})d\mathbf{V} \tag{3.10}$$

We use the following properties of the δ-function:

$$\int \delta(z)dz = 1 \quad \text{and}$$
$$\int r\delta(r-a)dr = a, \tag{3.11}$$

where 0 and a are within the respective integral limits. Knowing this, the calculations for each element of Q is left to the reader. After some algebra, the quadrupole moment turns out to be

$$Q_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \frac{q}{2}\left(a^2 - b^2\right) \tag{3.12}$$

This means the potential of the two charged concentric rings is approximately

$$\Phi(\mathbf{r}) \approx \frac{(a^2-b^2)}{16\pi\epsilon_0 r^5}(x^2+y^2-2z^2) \tag{3.13}$$

3.3 Potential by Taylor Expansion

Here, I will show that the electrostatic potential $\Phi(x,y,z)$ can be approximated by the average of the potentials at the positions perturbed by a small quantity $+/-a$ by doing a Taylor expansion.

This expansion is correct to the third order. First, the Taylor expansion around the $\Phi(x, y, z)$ perturbed in the positive x-component

$$\Phi(x+a, y, z) = \Phi(x, y, z) + \frac{\partial \Phi(x, y, z)}{\partial x} a + \frac{\partial^2 \Phi(x, y, z)}{\partial x^2} a^2 + \frac{\partial^3 \Phi(x, y, z)}{\partial x^3} a^3 + O(a^4) \tag{3.14}$$

Next, the same expansion around $\Phi(x - a, y, z)$:

$$\Phi(x-a, y, z) = \Phi(x, y, z) - \frac{\partial \Phi(x, y, z)}{\partial x} a + \frac{\partial^2 \Phi(x, y, z)}{\partial x^2} a^2 - \frac{\partial^3 \Phi(x, y, z)}{\partial x^3} a^3 + O(a^4) \tag{3.15}$$

When we add up these two equations, the odd powers of a cancel. This is the same for the y- and z-component. The a^2-term adds up to the Laplacian ∇^2. Assuming there is no charge within the radius a of (x, y, z), Laplace's equation holds:

$$\nabla^2 \Phi = 0 \tag{3.16}$$

and thus the a^2 term is zero, too. Therefore, the potential at (x, y, z) can be given by:

$$\begin{aligned}\Phi(x, y, z) = 1/6(&\Phi(x+a, y, z) + \Phi(x-a, y, z) + \Phi(x, y+a, z) \\ &+\Phi(x, y-a, z) + \Phi(x, y, z+a) + \Phi(x, y, z-a)) + O(a^4)\end{aligned} \tag{3.17}$$

Appendix A

Mathematical Tools

A.1 Partial integration

$$\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx \tag{A.1}$$

A.2 Vector analysis

Stokes' theorem

$$\oint_L \mathbf{M} \cdot d\mathbf{l} = \int_\mathbf{S} \mathrm{curl}\mathbf{M} \cdot d\mathbf{S} \tag{A.2}$$

Gauss' theorem

$$\oint_S \mathbf{M} \cdot d\mathbf{S} = \int_\mathbf{V} \mathrm{div}\mathbf{M} \cdot d\mathbf{V} \tag{A.3}$$

Computation of the curl

$$\mathrm{curl}\mathbf{M} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_i \hat{i} & h_2 \hat{j} & h_3 \hat{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_2} \\ h_1 M_1 & h_2 M_2 & h_3 M_3 \end{vmatrix} \tag{A.4}$$

h_i are the geometrical components that depend on the coordinate system. For a Carthesian coordinate system they are one. For the cylindrical sytem:

$$h_1 = 1$$
$$h_2 = r$$
$$h_3 = 1$$

For the spherical coordinate system:

$$h_1 = 1$$
$$h_2 = r$$
$$h_3 = r\sin\theta$$

Computation of the divergence

$$\mathrm{div}\mathbf{M} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial x_1}(h_2 h_3 M_1) + \frac{\partial}{\partial x_2}(h_1 h_3 M_2) + \frac{\partial}{\partial x_3}(h_1 h_2 M_3)\right) \tag{A.5}$$

With the same factors h_i depending on the coordinate system.

Relations between grad and div

$$\nabla \times \nabla \times \mathbf{M} = \nabla\nabla \cdot \mathbf{M} - \nabla^2 \mathbf{M} \tag{A.6}$$

Any vector can be written in terms of a vector potential and a scalar potential:

$$\mathbf{M} = \nabla\Phi + \nabla \times \mathbf{A} \tag{A.7}$$

A.3 Expansions

Taylor series

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2 f''(a)}{2!} + ... + \frac{x^n}{n!}(x-a)^n f^{(n)}(a) \tag{A.8}$$

A.4 Euler Formula

$$e^{i\phi} = \cos\phi + i\sin\phi \tag{A.9}$$

$$\begin{aligned}\sin x &= \frac{e^{ix}-e^{-ix}}{2i} &\text{(A.10)}\\ \cos x &= \frac{e^{ix}+e^{-ix}}{2} &\text{(A.11)}\\ \sinh x &= \frac{e^{x}-e^{-x}}{2} &\text{(A.12)}\\ \cosh x &= \frac{e^{x}+e^{-x}}{2} &\text{(A.13)}\end{aligned}$$

A.5 Trigonometry

$$\sin^2 x = \frac{1-\cos(2x)}{2} \qquad \text{(A.14)}$$

$$\begin{aligned}\cos(\phi-\theta) &= Re\left[e^{i(\phi-\theta)}\right] = Re\left[e^{i\phi}e^{-i\theta}\right]\\ &= Re\left[\cos\phi\cos\theta+\sin\phi\sin\theta+i(..)\right]\\ &= \cos\phi\cos\theta+\sin\phi\sin\theta &\text{(A.15)}\end{aligned}$$

Bibliography

[1] R. Bracewell. *The Fourier Transform and Its Applications*. McGraw-Hill Book Company, first edition, 1965.

[2] W.J. Duffin. *Electricity and Magnetism*. McGraw-Hill Book Company, fourth edition, 1990.

[3] J.D. Jackson. *Classical Electrodynamics*. John Wiley & Sons, Inc., third edition, 1999.